BOUNDS FOR MOMENT GENERATING FUNCTIONS AND EXTINCTION PROBABILITIES

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D E D I C A T E D

T O

M Y P A R E N T S

C E R T I F I C A T E

This is to certify that **Mr. Mohammad Vasim Khan** has completed his **M.Phil.** dissertation "BOUNDS FOR MOMENT GENERATING FUNCTIONS AND EXTINCTION PROBABILITIES" under my supervision.

Mr. Mohammad Vasim Khan is allowed to submit the work for the award of the **M.Phil.** degree in **STATISTICS** of the Aligarh Muslim University, Aligarh.

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P R E F A C E

Branching processes form an active area of research in the field of applied probability, physical and biological sciences and other fields.

The problem of obtaining bounds on the extinction time distribution, moments of extinction time distribution and probability of extinction of a Bienayme-Galton-Watson process has been considered during the last two decades by many authors. Heathcote and Seneta(1966) were the first to present bounds for ET and μ for subcritical processes with $g''(1) < \infty$. Likewise Erickson(1971), Harkness and Shantaram(1969), Kesten, Ney and Spitzer(1966) and Seneta(1967) have obtained bounds on moments of the extinction time distribution and the probability of extinction of a branching process. The asymptotic properties of subcritical Galton-Watson process have been obtained by Bagley(1982) and Seneta(1968).

It is practically impossible to present a review on all that has been done in this area and therefore an humble effort has been made to identify some of the problems that have been considered during the past few years and to review the relevent contributions that have been made. So far.

This dissertation consists of three chapters. Chapter I is devoted mainly to historical background of the branching processes and also it contains some basic concepts and results relevent to the subsequent chapters.

Chapter II deals with the fractional linear generating functions and their use in obtaining bounds for the p.g.f. These bounds have been used to obtain bounds for the extinction time distribution, moments of the extinction time distribution and the probability of extinction of the Bienayme-Galton-Watson branching process.

In Chapter III we present some recent results on asymptotic properties of subcritical Galton-Watson process.

I have tried my best to make the subject clear, understandable and stress is laid throughout on the explanation of fundamental concepts.

I have great pleasure in taking this opportunity to acknowledge my deep sense of gratitude to my supervisor Prof. Sirajur Rahman, Chairman, Department of Statistics and Operations Research, Aligarh Muslim University, Aligarh, for his valuable guidence, constant encouragement and providing me all the facilities for completion of this dissertation.

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C O N T E N T S

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C H A P T E R - I

CHAPTER-I

INTRODUCTION

1.1 Historical Background

Unit recently it was believed that the theory of branching processes seems first to have begun with the Galton-Watson Criticality Theorem (1873, 1874). However, the research work carried out by C.C.Hyde and E.Seneta (1972) established beyond any doubt the fact that its origin goes back to I.J.Bienayme.

I.J.Bienayme was born in Paris on 28 August 1796 and died there on either 19 or 20 October 1878. He joined the civil service in 1820 and was appointed general inspector of Finance in 1834. After the revolution of 1848 he retired and devoted all his time to scientific work.

An early work on the extinction of noble families in France, entitled " Memoire Sur la duree des families nobles de France" , was written by L.F.Benoiston de Chateauneuf (1775-1856) and first read in two of the meetings of " Memoires de 1 'Academie rovale des sciences morales et politiques de 1' Institute de France" .

Undoubtedly, this paper of de chateauneuf, among others stimulated I.J.Bienayne who treated the same problem mathematically in his paper " De la loi de multiplication et de La duree des families" read out on 29 March 1845(Kendall 1975).

Bienayme's paper appears as an appendix in Kendall's paper. Both title of the paper and its opening paragraph reveal the similarity in motivation between him and Galton. It also shows that the correct statement of the Criticality Theorem was knownto him:

" If ------ the mean of the number of male children who replace the number of male of the preceeding generation were less than unity, it would be easily realized that families are dying out due to the disappearance of the numbers of which they are composed. However, the analysis shows further that when this mean is equal to unity families tend to disappear, although less rapidly -------" (Quoted in Hyde and Seneta 1977, p.117).

In connection with Bienayme's methods, certain observations can be made. First, it is noticeable that he refers to a difference equation of the first order but of a degree equal to the maximum number of children. This is a reference to what would be written in the form:

$$q_{n+1} = f(q_n) \qquad \ldots \quad (1.1.1)$$

where q_n denotes the probability of extinction after n generation, and

$$f(s) = \Sigma_{k=0}^{\infty} p_k s^k,\ 0 \le s \le 1;\ p_k \ge 0,\ \Sigma_{k=0}^{\infty} p_k = 1$$

is the probability generating function for the number of 'sons' to a 'potential father'.

Second, Bienayme was aware of the fact that q_n will increase monatonically as n increases, and that it will converge to a limit $q \leq 1$ as n tends to infinity, which limit will satisfy the equation.

$$f(q) = q \qquad \ldots (1.1.2)$$

Bienayme's argument, so far, runs parallel to that of Watson, before a change in noticed.

The clue to the next section of Bienayme's argument lies in an at first mysterious remark in which he says that when m>1, then q is given by " the root of the equation(1.1.2) which is less than unity" (Kendall 1975, p.233). Bienayme's paper, too, states that a population is not allowed by the branching process to remain in a stationary state, contrary to what authors of life tables suppose in their calculation.

In 1873, the Swiss Mathematician, de Candolle, who did not hear of Bienayme's work, published his work, " Histoire des Sciences et des Savants Depuis Deux Sie eles" , in which he pointed to the possibility of a probabilistic interpretation for the phenomenon of the extinction of a large number of noble families. In the same year, F.Galton gave the problem a precise formulation as problem 4001, which was published in the 'Educational Times', and in which he says:

" A large nation, of whom we will only concern ourselves with the adult males, N in number and who each bear separate surnames, colonise a district. Their law of population is such that, in each generation, a_0 percent of the adult males have no male children who reach adult life; a_1 have one such male child; a_2 have two; and so on upto a_5 who have five. Find (1) what proportion of the surnames will have become extinct after r generations; and (2) how many instances there will be of the same surname being held by n persons" .

Galton, then, turned to his friend, H.W.Watson, who transformed the problem into one of iteration of generating function $f(s) = \Sigma_{j=0}^{\infty} p_j s^j$, $p_j = a_j/100$, i.e. the probability of a father begetting j male children reaching adult life, then Watson, in 1874, defines a sequence recursively by $f_r = f_{r-1} \circ f$. His conclusion is that the answer to the first question is the term independent of s in $f_r(s)$ and gives the number of surnames with k representatives in the rth generation as the coefficient of s^k in $f_r(s)$ multiplied by N. However, Watson's solution contains an algebraic oversight and he incorrectly concludes that each family will eventually die out with probability one. If Watson had read Schroder's work on functional interation in 'Mathematische Annalen'(1871), as it was suggested by Kendall (1966), he would have experimented with linear fractional generating functions anddiscovered that his last conclusion was wrong and lacked due deliberation.

Consequently, another half a century passed before the correct extinction probability was known, (for Bienayme's work came to light only in 1972).

The Galton-Watson process seems to have been neglected for a long time. Only in 1922 did R.A. Fisher touch upon the topic in a genetical context and followed it up in 1930 to study random variations in frequencies of genes. By that time, and in 1927, J.B.S. Haldane had applied the model to genetics and roughly sketched a correct answer of the Criticality Theorem, namely that essentially the extinction probability is one exactly when the mean $m = f'(1) \leq 1$.

In 1929, the same problem was independently treated by the Danish Erlang in the " Matematisk Tidsskrift" . Erlang's treatment of the problem shows that he realized as Watson had not, that equation (1.1.2) can have two roots in the relevent interval $[0,1]$ and that in fact there will be one root in $[0,1]$ in addition to the root $q = 1$ if and only if the expected number of sons per parent, m is greater than unity. From some remarks of Steffensen, included in his " Deux problems due calcul des probabilities" (1933), it is reasonable to believe that Erlang, before his death, conjectured what is in fact the basic theorem of the subject; it is always the smallest root of (1.1.2) which is the appropriate one; thus extinction is almost certain for subcritical populations with $m<1$ and for critical population with $m=1$, but there is always a positive chance of survival for supereritical populations with $1<m<\infty$.

A clear and detailed proof of this theorem was made by J.F.Steffensen in 1930 and 1933. Commenting on Steffensen's effort, W.P.Elderton remarked that the probabilities p_k might in practice prove to be in geometric progression. Steffensen, in his turn, quickly realized that if we put

$p_o = \alpha$, $p_k = (1-\alpha)(1-\beta)\ \beta^{k-1}$ $(k = 1, 2 \ldots\ldots)$ then f will be a linear fractional function and the iterations can be made explicit (Kendall 1966, p.389).

After reading Steffensen's article in the " Matematisk Tidskrift" , A.J.Lotka(1931) applied the branching process theorem to the data contained in the 1920 United States census of white male obtaining $q = 0.88$ as the probability of the termination of the male line of descent from a new born male.

Due mainly to the efforts of D.Hawkins and S.Ulam, T.E. Harris and A.M.Yaglom, the final solution to the Galton-Watson process was successfully evolved between 1944 and 1950.

More details concerning the historical development of branching processes can be found in Harris(1963), Kendall(1966, 1975), Jagers(1975) and Hyde and Seneta (1972, 1977).

1.2 Markov Chain

The stochastic process $(Z_n,\ n = 0, 1, 2, \ldots\ldots)$ is called a Markov Chain if, for $j, k, j_1, j_2, \ldots\ j_{n-1} \in N$(or any subset of the set of all integers I),

$$P(Z_n = k / Z_{n-1} = j,\ Z_{n-2} = j_{n-1} \ldots\ldots\ldots,\ Z_1 = j_2,\ Z_o = j_1)$$
$$= P(Z_n = k \mid Z_{n-1} = j) = P_{jk}\ \text{(say)}$$

whenever the first member is defined.

The probability of Zn being in state K given that Z_{n-1} is in state j is called one step transition probability and denoted by p_{jk}.

The transition probability may or may not be independent of the time no, however, which is dependent of n, the Markov chain is said to be homogeneous (or to have stationary transition probabilities).

1.3 Galton-Watson Branching Process

Let the random variables Z_0, Z_1, Z_2 ----- denote the size of (or the no. of objects in) the 0th, Ist, 2nd, ...generations respectively. Let the probability that an object(irrespective of the generation to which it belongs) generates k-similar objects be denoted by p_k, where $p_k \geq 0$, $k = 0,1,2,......$; $\Sigma_{k=0}^{\infty} p_k = 1$.

The sequence $\{Z_n, n=0, 1,2, ...\}$ constitutes a Bienayme-Galton-Watson(BGW) or simply a Galton Watson(GW) branching process with offspring distribution $\{p_k\}$.

Formally $\{Z_n, n=0,1,2,.....\}$ is a time homogeneous Markov Chain with state space N and with transition probabilities

$$P_{ij} = P(Z_{n+1} = j \mid Z_n = i) = \begin{cases} p_j^{*i} & \text{if } i \geq 1,\ j \geq 0 \\ \delta_{oj} & \text{if } i = 0, j \geq 0 \end{cases} \qquad ...(1.3.1)$$

δ_{ij} being the Kronecker delta and $\{p_j^{*i}, j=0,1,2,\ldots\}$ being the i-fold convolution of $\{p_j, j=0,1,2,\ldots\ldots\}$. The transition probabilities satisfy

$$p_{oo} = 1 \text{ and } \Sigma_{j=0}^{\infty} p_{ij}s^j = (\Sigma_{j=0}^{\infty} p_{1j}\, s^j)^i,\ i\geq 1,\ 0\leq s\leq 1.$$

From the definition of $\{Z_n\}$ as a Markov chain with a given transition function, we know from general considerations (the Kolmogorov theorem) that there is a probability space (Ω, F, P) on which $\{Z\ (w);\ n\geq 0\}$ are defined, and have the distributions determined by (1.3.1).

By extinction we mean the event that the random sequence $\{Z_n\}$ consists of zeros for all but a finite number of values of n.

Since Z_n is integer-valued, extinction is also the event that $Z_n \longrightarrow 0$. Moreover, since $P(Z_{n+1}=0 \mid Z_n = 0) = 1$, we have the equalities

$$\left.\begin{aligned} P(Z_n \longrightarrow 0) &= P(Z_n=0 \text{ for some } n) \\ &= P[(Z_1=0)\ U(Z_2=0)U\ \ldots\ldots] \\ &= \lim_{n\to\infty} P[(Z_1=0)\ U\ \ldots .U(Z_n=0)] \\ &= \lim P(Z_n=0) = \lim f_n(0). \end{aligned}\right\} \quad \ldots\ (1.3.2)$$

It is obvious that $f_n(0)$ is a nondecreasing function of n. Let q be the probability of extinction i.e.

$$q = P(Z_n \longrightarrow 0) = \lim f_n(0) \ \ldots\ldots\ (1.3.3)$$

Throughout we shall assume that;

(1) the process starts with a single ancestor, i.e. $Z_0=1$;

(2) F is non-degenerate, i.e. $p_k<1$ for all k, and that $P(Z_1=0) < 1$;

(3) $p_0 + p_1 < 1$

It is clear from (1.3.1) that if $Z_n=0$, then with prob 1, $Z_{n+k}=0$ for all $k \geq 0$. Thus 0 is an absorbing state, and reaching 0 is same as the process being extinct. All other states 1,2,... are transient, that is, $Z_n \longrightarrow \infty$ a.s. on E^c where

$E: = \{Z_n=0 \text{ eventually}\}$ is the set of extinction.

1.4 Probability Generating Function and Extinction Probability

An important and useful tool in deriving properties of the BGW branching process and of more sophisticated branching processes is the probability generating function (p.g.f.), and it will be the main object of attention in this dissertation.

As we have mentioned in section (1.1), Watson noticed the very important fact that the p.g.f. for Z_n is the n^{th} functional iterate of the p.g.f. for Z_1. That is, if

$$f(s) = \Sigma_{k=0}^{\infty} p_k \, s^k, \quad |s| \leq 1 \qquad \ldots \qquad (1.4.1)$$

is the p.g.f. for Z_1, and

$$f_n(s) = \Sigma_{k=0}^{\infty} P(Z_n=k)s^k \qquad \ldots \qquad (1.4.2)$$

is the p.g.f. for Z_n, n=0,1,2,....., with $f_0(s) = s$ and $f_1(s) = f(s)$, then

$$f_n(s) = \underbrace{f(f(\ldots\ldots(f(s))}_{n \text{ times}} \ldots\ldots)).$$

Further,

$$f_n(s) = f(f_{n-1}(s)) = f_{n-1}(f(s)), \; n=1,2,\ldots\ldots \quad (1.4.3).$$

In particular, setting $s=0$ in (1.4.2), $P(Z_n=0) = f_n(0)$. Also if $m = f'(1) = E(Z_1) < \infty$, then $f_n'(1) = E(Z_n) = m^n$, and if $\sigma^2 = f''(1) + f'(1) - (f'(1))^2 = Var(Z_1) < \infty$, then

$$Var(Z_n) = \begin{cases} \sigma^2 m^{n-1}(m^n-1)/(m-1) & \text{if } m \neq 1 \\ n\sigma^2 & \text{if } m = 1 \end{cases}$$

Thus the variance of Z_n increases or decreases almost geometrically if $m>1$ or $m<1$ and linearly if $m=1$.

The process is called subcritical, critical, supercritical or explosive, depending on whether $m<1$, $m=1$, $1<m<\infty$, or $m = \infty$.

Let s be real. From the definition of f as a power series with non-negative coefficients $\{p_k\}$ adding to 1, and with $p_o+p_1<1$, we see at once that,

(1) f is strictly convex and increasing in $[0,1]$;

(2) $f(o) = p_o$, $f(1) = 1$;

(3) if $m \leq 1$ then $f(s) > s$ for $s \in [0,1)$;

(4) if $m>1$ then $f(s)=s$ has a unique root in $[0,1)$.

Now, let q denote the probability of eventual extinction of the BGW branching process. Then

$$q = P(\lim_{n\to\infty} Z_n=0) = \lim_{n\to\infty} P(Z_n=0) = \lim_{n\to\infty} f_n(o) .$$

Let q be the smallest root of $f(s)=s$ for $s \in [0,1]$. Then (1) - (4) imply that there is such a root, and furthermore,

<u>Lemma</u> 1.4.1 If $m \leq 1$ then $q = 1$, $P(Z_n \longrightarrow \infty) = 0$ while if $m>1$ then $q < 1$, $P(Z_n \longrightarrow \infty) > 0$.

1.5 Some Basic Theorems on BGW Branching Process

Our purpose of this section is to give some basic but useful limit theorems about Z_n, that enable us to study the behaviour of Z_n, when n is large. It has already been seen that the sequence $\{Z_n\}$ either goes to ∞ or goes to 0, it does not remain positive and bounded, even in case $m=1$ (for the proofs of these theorems we refer to Athreya and Ney(1972) Ch.I, and Asmussen and Hering (1983), Ch.III).

It is imperative at this stage to define various modes of convergence of a sequence $\{X_n\}$, $n=1,2,\ldots,$ to the random variable X,

(1) Convergence in probability means for each $\epsilon>0$, we have

$$\lim_{n\to\infty} P\{|X_n - X| > \epsilon\} = 0;$$

(2) Convergence in mean square means

$$\lim_{n\to\infty} E|X_n - X|^2 = 0;$$

(3) Convergence with probability 1 means with probability 1 the $\lim_{n\to\infty} X_n$ exists and is equal to X, i.e.,

$$\lim_{n\to\infty} X_n = X \text{ a.s.}$$

For subcritical process, we have

Theorem 1.5.1

If $0 < m < 1$, then

$$E(s^{Z_n} \mid Z_n > 0) = \frac{f_n(s) - f_n(0)}{1 - f_n(0)} \longrightarrow k(s) \text{ as } n \to \infty ,$$

$s \in [0,1]$, where $k(s)$ is the unique p.g.f. solution of

$$1-k(f(s)) = m(1-k(s)); \; k(0) = 0 \qquad \ldots \qquad (1.5.1)$$

The mean of this distribution is $k'(1) = \lim_{n \to \infty} \dfrac{m^n}{1-f_n(0)} = \mu(f)$ say, $k'(1) > 1$.

Note that $P(Z_n > 0) = 1-f_n(0) \sim \dfrac{m^n}{\mu(f)}$.

If $f''(1) < \infty$, then $k'(1)$ and $K''(1)$ are finite, we can differentiate (1.5.1) twice at $s=1$ to get

$$k''(1) = \frac{\mu(f).f''(1)}{m(1-m)}$$

For critical branching process, we have

Lemma 1.5.1

If $m=1$ and $\sigma^2 = \text{Var } Z_1 < \infty$, then

$$\lim_{n \to \infty} \frac{1}{n} \left[\frac{1}{1-f_n(s)} - \frac{1}{1-s} \right] = \frac{\sigma^2}{2} ,$$

uniformly for $0 \leq s < 1$.

Since $P(Z_n > 0) = 1-f_n(0)$, thus we obtain (setting $s=0$ in lemma 1.5.1) the following estimate of the rate of convergence to 0 (first proved by Kolmogorov (1938)).

Theorem 1.5.2

If $m=1$ and $\sigma^2 < \infty$, then as $n \longrightarrow \infty$

$$1 - f_n(0) = P(Z_n > 0) \sim \frac{2}{n\sigma^2}.$$

Since every non-zero state is transient, then either $Z_n \longrightarrow 0$ or $Z_n \longrightarrow \infty$. In critical case $Z_n \longrightarrow 0$ w.p. 1. On the other hand the limit probabilities of the sequence of conditional distributions of $\{Z_n | Z_n > 0\}$ are zero, and hence this process is divergent to ∞. An idea as to the rate of divergence is given by a simple moment calculation;

$$1 = E\, Z_n = E(Z_n | Z_n > 0) . P(Z_n > 0) + 0 . P(Z_n = 0)$$

implying that

$$E(Z_n | Z_n > 0) = \frac{1}{P(Z_n > 0)} \sim \frac{n f''(1)}{2} \quad \text{(by Theorem 1.5.2)}$$

i.e., the mean of the conditional process is growing at the rate n. Therefore, it is reasonable to cut the process down by a factor n. This leads to the following:

Theorem 1.5.3

If $p_1 \neq 1$, $m=1$ and $\mu = \frac{1}{2} f''(1) < \infty$,

then

$$\lim_{n \to \infty} P\left(\frac{Z_n}{n} > \lambda \mid Z_0 = 1, Z_n > 0 \right) = e^{-\lambda/\mu}, \quad \lambda \geq 0$$

This theorem was originally proved by Yoglom (1947) under a third moment assumption.

Another variable of interest is $W_n = X_n/m^n$, n=0,1,2,... $\{W_n\}$ forms a Markov chain. We have $E\{W_n\} = 1$ and for m >1,

$$E\{W_n^2\} = \frac{1}{m^{2n}} E(X_n^2) = \frac{1}{m^{2n}}\left\{m^{2n} + \frac{m^{n-1}(m^n-1)\sigma^2}{(m-1)}\right\}$$

Dividing both qides of $E\left\{X_{n+r} \mid X_n\right\} = X_n m^r$ by m^{n+r}, we get

$$E\left\{W_{n+r} \mid W_n\right\} = W_n \qquad \ldots \qquad (1.5.2)$$

and since $\left\{W_n\right\}$ is also a Markov chain.

$$E\left\{W_{n+r} \mid W_n, W_{n-1}, \ldots\ldots, W_o\right\} = E\left\{W_{n+r} \mid W_n\right\} = W_n$$

Theorem 1.5.4

If $0 < m < \infty$, then $\left\{W_n, F_n, n = 0,1,2,\ldots\ldots\right\}$, where F_n is a σ-algebra generated by $Z_0, Z_1, \ldots\ldots, Z_n$, form a martingale. Furthermore, since $W_n \geq 0$, there exists a random variable W such that

$$\lim_{n \to \infty} W_n = W \text{ a.s.}$$

From (1.5.3) we see that $Z_n \backsim m^n W$, this means that the population increases at a geometric rate, in accordance with Malthusian law of growth. Although, Theorem 1.5.4 gives an interesting result under a very week hypothesis it tell us nothing about W. It could be meaningful, if at all, only when m>1 and moreover when $\sigma^2 < \infty$, We can then assert that W is non-degenerate.

Theorem 1.5.5

If $m>1$, $\sigma^2<\infty$, and $Z_0=1$, then

(i) $\lim_{n\to\infty} E(W_n-W)^2 = 0$;

(ii) $EW=1$, $\text{Var } W = \sigma^2/(m^2-m)$;

(iii) $P(W=0) = q = P(Z_n=0 \text{ for some } n)$.

The mean square convergence of W_n to W in the above theorem was first established by Harris (1948).

An application of Laplace-Stieltjes transform reveals something more about the random variable W.

Let $\phi_n(U) = E(e^{-uW_n})$ and $\phi(u) = E(e^{-uW})$, $u\geq 0$ be the Laplace-Stieltjes transform of the distribution function of W_n and W respectively. Then writing $t_n = e^{-u/m^n}$,

$$\phi_n(u) = E(e^{-uW_n}) = E(t_n^{Z_n}) = f_n(t_n)$$

and

$$\phi_{n+1}(mu) = E(e^{-muZ_{n+1}/m^{n+1}}) = E(t_n^{Z_{n+1}})$$

$$= f_{n+1}(t_n) = f(f_n(t_n)) \text{ by } (1.4.3)$$

i.e. $\phi_{n+1}(mu) = f(\phi_n(u))$.

Since the random variables W_n converge in probability to W, their distributions converge to that of W and $\phi_n(u) \longrightarrow \phi(u)$ when $u\geq 0$. Therefore letting $n \to \infty$ in the last equation we find that the Laplace-Stieltjes transform of W satisfies the fundamental equation

$$\phi(mu) = f(\phi(u)), \quad u \geq 0$$

$$\text{with } \phi(0+) = 1 \qquad \ldots. \qquad (1.5.4)$$

The solution of (1.5.4) is unique upto a scale factor, that is, if ϕ_1 and ϕ_2 are two solutions, then there is a constant c such that $0<c<\infty$ and $\phi_1(u) \equiv \phi_2(cu)$. (Asmussen and Hering (1983), p.84).

The distribution function of W has a probability of mass q at the origin and is absolutely continuous on $(0,\infty)$ with continuous positive density if $m>1$ and $\sigma^2<\infty$ (Harris(1963), p. 16).

1.6 Bellman-Harris Branching Processes

So far we have been considering branching process $\{Z_n; n\geq 0\}$ in discrete time: an object after one unit of time produces, similar objects according to off-spring distribution $\{p_k\}$. Now we proceed to consider a generalization such that the life times of objects are i.i.d. random variables. Instead of the process $\{Z_n, n\geq 0\}$ we shall consider the process $\{Z(t), t\geq 0\}$, where $Z(t)$ equals the no. of objects (or particles, individuals, organisms) at time t. The process $\{Z(t), t\geq 0\}$ may or may not be Markeovian. If the lifetimes of objects are exponential random variables, then the process $\{Z(t), t\geq 0\}$ is a Markovian process. In this section however, we consider the general case where the lifetimes of objects do not necessarily have exponential distributions.

Suppose that an object (ancestor) at time t=0 initiates the process. At the end of its lifetime, it produces a random number of descendents according to the off-spring distribution

$\{p_k\}$ (with p.g.f.h(s)). We assume that these descendents act independently of each other and that at the end of its lifetime, each one produces its own offspring with the same distribution $\{p_k\}$, and that the process continues as long as objects are present. The lifetimes T's are independent random variables with distribution function $G(t)=P(T\leq t)$; object production is independent of the present state or past history of the process; and the lifetimes and object production variables are independent.

The stochastic process $\{Z(t);\ t\geq 0\}$ is known as an age-dependent or general time branching process. This process is sometimes also reffered to as Bellman-Harris process, after Bellman and Harris first considered it in 1948.

In thes works reviewed in this section, we find that as before, generating function of the process plays the key role in the analysis and the work centers around an integral equation satisfied by the generating function of Z(t),

$$F(s,t) = \Sigma_{k=0}^{\infty} P[Z(t) = K].S^k \qquad \ldots \qquad (1.6.1)$$

To find $P[Z(t)=k]$, we shall condition on the life time T at which the ancestor dies bearing i offspring. We have

$$\begin{aligned} P[Z(t)=k] &= \int_0^{\infty} P[Z(t) = k \mid T = u].dG(u) \\ &= \int_0^{t} P[Z(t) = k \mid T = u].d\,G(u) + \int_t^{\infty} P[Z(t)=k \mid T=u]\ dG(u) \end{aligned}$$

In case of second term, u>t. Given that T=U, the number of objects at time t is then still 1(the ancestor), and the second term yields $\delta_{1k}[1-G(t)]$, where δ_{jk} is the Kronecker delta. In case of first term $u \leq t$, the ancestor dies at time $u \leq t$, leaving i direct descendents: the probability of this is $p_i\, dG(u)$, and further these i descendants(who independently initiate processes at time u) leave k objects in the remaining time t-u: the probability of this is $\Sigma_{i=0}^{\infty} p_i P^{*i}[Z(t-u)=k]$, where P^{*i} is the i-fold convolution of P. Thus

$$P[Z(t)=K]=[1-G(t)].\delta_{1k}+\int_0^t dG(t).\Sigma_{i=0}^{\infty} p_i.P^{*i}\ [Z(t-u)=K]$$

Now multiplying throughout by s^k, summing over k, then the generating function F(s,t) of Bellaman-Harris process satisfies, the integral equation.

$$F(s,t)=s[1-G(t)]\ +\int_0^t h[F(s,t-u].dG(u).\ |S|\leq 1$$

The integral equation cannot easily be solved in the general case. However, in particular when $G'(t)=be^{-bt}$, we can see that this integral equation reduces to

$$F(s,t)=se^{-bt}+be^{-bt}\int_0^t h[F(s,u)].e^{bu}\,du,$$

Whence

$$\frac{\partial F}{\partial t}=b[h\{F(s,t)\}-F(s,t)].$$

We conclude this section with the following observation about the expectation M(t) = E(Z(t)). Its asymptotic behaviour will, however, be studied in chapter III.

The expectation $M(t)$ of a Bellman-Harris process $\{Z(t), t\geq 0, Z_0=1\}$ satisfies the integral equation.

$$M(t) = [1-G(t)] + m\int_0^t M(t-u).dG(u) \qquad \ldots \quad (1.6.2)$$

where $m = h'(1)$ is the mean of the offspring distribution. Further, if $m=1$, then $M(t)=1$ is a solution of (1.6.2).

To prove (1.6.2) we shall condition on the lifetime T of the ancestor, to get

$$\begin{aligned} M(t) &= E[Z(t)] = \int_0^\infty E[Z(t)\mid T = u].dG(u) \\ &= \int_0^t E[Z(t)\mid T = u]dG(u) + \int_t^\infty E[Z(t)\mid T=u].dG(u) \end{aligned}$$

If $u>t$, then the number of objects at time t is still 1, then $E\{Z(t)\mid T=u\} = E(Z_0) = 1$.

If $u\leq t$, then the ancestor lives for time u at the end of which, it leaves i off-springs with probability p_i, each of these offspring initiates a process, the number of objects of such a process having the same distribution as $Z(t-u)$. Thus for $u\leq t$.

$$E[Z(t)\mid T = u] = \Sigma_{i=0}^{\infty} ip_i \; E\{Z(t-u)\} = m.M(t-u).$$

Then

$$M(t) = m\int_0^t M(t-u)\; dG(u) + \int_t^\infty dG(u)$$

Thus (1.6.2) follows, when $m=1$, then the solution of (1.6.2) is obviously $M(t) = 1$.

C H A P T E R - II

CHAPTER - II

Fractional Linear Generating Functions and Bounds on the Extinction - Time Distribution of Branching Process:

2.1 Introduction

Let T denote the extinction time of the Galton-Watson process, i.e. $T = K \iff Z_{k-1} > 0,\ Z_k = 0$.

The problem of obtaining bounds on the extinction time distribution of Bienayme - Galton-Watson branching process has been considered during the last two decades by many authors. Heathcate and Senata (1966) were the first to present bounds for ET and μ for subcritical processes with $g''(1) < \infty$. Under very general conditions (see Senata 1968a) as $m \longrightarrow 1^-$ their lower bound for ET converges to a finite limit (although $ET \longrightarrow \infty$), and their upper bound grows at a rate proportional to $(1-m)^{-1}$, which is exponentially faster than the actual rate.

There is a large class of p.g.f.'s for which these bounds are inapplicable and in general it is difficult to verify whether the bounds apply to a given p.g.f.. Seneta (1967 b), and Pollak (1969) have also derived bounds for the extinction time distribution when g has the Poisson form, which has been used in genetic applications of branching processes since Fisher (1930 a). Pollak(1971) has also considered the problem of deriving bounds for ET and μ. His bounds apply to the p.g.f.'s which can be shown to satisfy a certain inequality involving the first three derivatives of a p.g.f. evaluated at one.

The method of bounding a p.g.f. with $g''(1) < \infty$ by two fractional linear generating functions is used by Agresti (1974) to derive bounds for the extinction time distribution of the BGW branching process and in fact for the special case of the Poisson p.g.f., he has obtained the best bounding fractional linear generating function. Hwang and Wang(1979) has provided, under weak conditions a best lower and best upper bounding fractional linear generating function for any p.g.f. when they have the same means. Their bounds can be used to obtain bounds for the expectation and the percentiles of the extinction time distribution of a BGW branching process, and other parameters of interest.

Evans(1978) has provided an upper bound for the mean of the associated Yoglom limit $K(s)$. This bound is attained if and only if the p.g.f. of the process is linear, i.e. $g(s) = 1-m+ms$, $0 < m < 1$.

In this chapter an attempt has been made to present some results whose bounds for the p.g.f. and the extinction time distribution of the BGW branching process using an approach similar to that of Agresti(1974) and Hwang and Wang(1979) have been obtained. We first define the fractional linear generating function & examine some of its properties.

2.2 Fractional Linear Generating Functions

Only very few examples are known, for which $g_n(s)$, the n-fold convolution of a p.g.f. $g(s)$ has been calculated explicitly. Perhaps the most interesting one are the fractional linear generating functions (f.l.g.f.'s) which will be discussed in this chapter.

Definition 2.2.1

A p.g.f. with the following form:

$$f(m,c;s) = \Sigma_{j=0}^{\infty} p_j s^j = 1-m(1-c) + m(1-c)^2 s/(1-cs) \quad \ldots \quad (2.2.1)$$

where $0 \leq s \leq 1$ and $m(1-c) \leq 1$ will be called a fractional linear generating function with mean m.

The f.l.g.f., $f(m,c;s)$ is linear and the distribution reduces to a Bernoulli trials with $p_0=1-m$ and $p_1=m$ if $c=0$, and it reduces to a geometric distribution with $p_j= (1-c)c^{j-1}$, $j \geq 1$ when $m(1-c)=1$.

If $m \neq 1$, the equation $f(m,c;s) = s$ has two distinct non-negative solutions 1 and $s_0= [1-m(1-c)]/c = s_0(m,c)$ (say). If $m<1$, $s_0>1$, if $m>1$, $s_0 < 1$, and $s_0=q$; $s_0=1$ if and only if $m=1$, where q is the probability of eventual extinction.

The n^{th} iterate of $f(m,c;s)$ can be given explicitly in terms of m and s_0. If $m \neq 1$, then

$$\frac{f(m,c;s) - s_0}{f(m,c;s) - 1} = \frac{1}{m} \cdot \frac{s-s_0}{s-1}$$

Iterating this we get,

$$\frac{f_n(m,c;s) - s_0}{f_n(m,c;s) - 1} = \frac{1}{m^n} \cdot \frac{s-s_0}{s-1}$$

which can be solved to yield

$$\text{(i)}\ (m \neq 1)\ f_n(m,c;s) = 1 - m^n\left(\frac{1-s_0}{m^n - s_0}\right) + m^n\left(\frac{1-s_0}{m^n - s_0}\right)^2 S \Big/ \left(1 - \frac{m^n - 1}{m^n - s_0} S\right)$$

$$= f(m^n, C_n ; s),$$

which is also a f.l.g.f. with $C_n = (m^n - 1)/(m^n - s_o)$.

$$\text{(ii)}\ (m=1)\ f_n(m,c;s) = [nc-\{(n+1)c-1\}s] \,/\, [1+(n-1)c-ncs]$$

$$= f(1, c_n ; s)$$

again a f.l.g.f. with $c_n = nc/[1+(n-1)c]$ ... (2.2.2.)

The f.l.g.f.'s have occured in work on functional iteration (Schroeder(1871)). They proved useful in the study of the extinction probability of male line of descent (Lolka (1931), (1939)), and they play an important role in connection with the problem of embeddability of a BGW branching processes in a continuous time branching processes (Karlin and McGregor (1968 a, b)).

2.3 Moments of the Extinction Time Distribution and Bounds on Moments and Probability of Extinction of a Branching Process

For moments of the extinction time distribution, R. V. Erickson in 1971 stated that, let Z_n be the number of individuals in the n^{th} generation of a reproducing system so that Z_n is a Galton-Watson process, and let $\tau = \inf\{n \mid Z_n = 0\}$ be the extinction time for the process. It is well known that $m = EZ_1 > 1$ iff $P(\tau < \infty) < 1$ and that if $m < 1$ then $P(\tau > n) \leqq m^n$. Thus, in the non-critical case, $m \neq 1$, either $E\tau^\alpha < \infty$ for no $\alpha > 0$ or all $\alpha > 0$. There is a drastic difference in the critical case.

If $m = 1$ (see Kesten, Ney, Spitzer(1966)) than $P(\tau > n) \backsim \sigma^2/2n$, when $\sigma^2 = \mathrm{Var}\, Z_1 < \infty$ and in this case $E\tau^\alpha < \infty$ iff $0<\alpha<1$. It is pointed out in the above reference that $nP(\tau > n) \longrightarrow 0$ if $\sigma^2 = \infty$. This fact is what led us to ask about higher moments of τ when Z_1 has worse behaviour and we have the following negative answer.

<u>Theorem</u> 2.3.1 Let $m = 1$. If $EZ_1^{1+\alpha} < \infty$ then $E\tau^\beta = \infty$ for all $\beta > 1/\alpha$, $0 < \alpha < 1$, we have been unable to determine the validity of the next.

<u>Assertion</u> Let $m = 1$, If $EZ_1^{1+\alpha} < \infty$ then $E\tau^\beta < \infty$ for all $\beta < 1/\alpha$, $0 < \alpha < 1$.

However, if stronger restriction than $EZ_1^{1+\alpha} = \infty$ are placed on the distribution of Z_1 results of the above type do hold. To make these precise we need additional notation.

Let $f(s) = \Sigma_0^{\infty} s^n P(Z_1=n)$ be the probability generating function of Z_1, so that the p.g.f. of Z_n is f_n, the n^{th} functional iterate of f, i.e. $f_n = f \circ f_{n-1}$, $f_0 =$ identity. Then $P(\tau > n) = 1-f_n(o)$ and $m = f'(1-)$.

From now on assume $m = 1$.

We introduce the following conditions on f, where K is some finite positive constant.

A_β : $f(s)-s \leqq K(1-s)^{1+\beta}$ for $s_0 \leqq s \leqq 1$, some $s_0 < 1$.

B_β : $f(s)-s \geqq K(1-s)^{1+\beta}$ for $s_0 \leqq s \leqq 1$, some $s_0 < 1$.

It is easy to show that $EZ_1^{1+\beta} < \infty$ implies A_β, (see e.g. Loeve (1963) page 199). Further, A_β implies $EZ_1^{1+\alpha} < \infty$ for each $\alpha < \beta$, $o < \beta < 1$, while B_β implies $EZ_1^{1+\beta} = \infty$, $o < \beta < 1$. We know of no moment condition which implies B_β, but we do have the following sufficient condition (Theorem 2.3.7): If $P(Z_1 > x)$ is asymptotic to $x^{-1-\alpha} L(x)$, where L is slowly varying, then B_β holds for each $\beta > \alpha$.

<u>Theorem</u> 2.3.2 : Let $o < \beta < 1$. Condition B_β implies $EZ_1^{1+\beta} = \infty$ and $E\tau^{\gamma} < \infty$ for all $\gamma < 1/\beta$. Since $E\tau^{\gamma} < \infty$ iff $\Sigma n^{\gamma-1}(1-f_n(o)) < \infty$, Theorems (2.3.1) and (2.3.2) are corrolories of

<u>Theorem</u> 2.3.3 : Let $\beta > 0$ and fix ν_0 in $[0,1]$. Then A_β implies $n^{\gamma}(1-f_n(\nu_0)) \longrightarrow 0$ for all $\gamma < 1/\beta$. Let X be a nonnegative r.v. with df F and Laplace transform $\phi(\lambda) = \int_{0-}^{\infty} e^{-\lambda x} F(dx)$. Set $\mu_x = EX^x$ and introduce inducively the notation

$$F_o := F,\ F_{n+1}(x) := \int_o^x \left[\frac{\mu n}{n!} - F_n(y) \right] dy$$

for each n such that $\mu_n < \infty$.

The following theorem (perhaps well known) gives a representation for μ_n and $\emptyset$ in terms of F_n.

Theorem 2.3.4 :- Assume $\mu_m < \infty$, m a non-negative integer. Then for n = 0, 1, ..., m

(a) $F_n(x) \uparrow \mu_n/n!$ as $x \uparrow \infty$;

(b) $\mu_{n+\alpha} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+\alpha-k)} \int_0^\infty x^{n+\alpha+k-1} \left[\frac{\mu k}{K!} - F_k(x) \right] dx,$

k=0, 1, ..., n, $\alpha \geqq 0$, $n+\alpha-k>0$, in the sense that if one side is finite so is the order;

(c) $(-1/\lambda)^{n+1} [\emptyset(\lambda) - \Sigma_o^n \mu_k(-\lambda)^k/K!] = \int_0^\infty e^{-\lambda x}[\frac{\mu_n}{n!} - F_n(x)]dx$

Theorem 2.3.5

(a) For $o<\alpha<1$, $\mu_{m+\alpha} < \infty$ iff for some (and thus all) $c > o \int_o^c \eta_{m,\alpha} < \infty$, and then (whether finite or not)

$$\mu_{m+\alpha} = \frac{\Gamma(m+\alpha+1)}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^\infty \eta_{m,\alpha}$$

(b) $\nu_m := EX^m \operatorname{Log} x < \infty$ iff for some (and all) $C > o \int_0^c$, $\eta_{mo} < \infty$ and then (whether finite or not)

$$\nu_m = m! \int_o^\infty \eta_{m,o}$$

Theorem 2.3.6 :- Let X be a nonnegative r.v. with df F and Laplace transformation $\emptyset$. Assume that $o<\alpha<1$ and that m is the nonnegative integer such that $\mu_m < \infty = \mu_{m+1}$. The following are equivalent (where $k = 0, 1, \ldots, m$, and $x = 1/\lambda \longrightarrow \infty$:

$$(a_k)\ \frac{\mu_k}{k!} - F_k(x) \sim \frac{\Gamma(m+\alpha-k)}{\Gamma(m+\alpha)}\ x^{-m-\alpha+k}\ .\ L(x)$$

$$(a_{m+1}) F_{m+1}(x) \sim \frac{\Gamma(\alpha)}{(1-\alpha)\ \Gamma(m+\alpha)}\ x^{1-\alpha}\ L\ (x)$$

$$(b_k)\ \eta_{k,\alpha-1}\ (\lambda)\quad \frac{\Gamma(1-\alpha)\ \Gamma(\alpha)}{\Gamma(m+\alpha)}\ \lambda^{m-k}\quad L\ (1/\lambda)\ .$$

Theorem 2.3.7 :- If $P(Z_1 > x) \sim x^{-1-x}\ L(x)$, where L is slowly varying, then f satisfies condition B_β for all $\beta > \alpha$, whence $E\ \tau^\gamma < \infty$ for all $\gamma < 1/\alpha$ and $EZ_1^\beta = \infty$ for all $\beta > \alpha$.

Remarks : It was hoped that $\int_0^\infty \eta_{1,\alpha-1} = \infty$ (iff $EZ_1^{1+\alpha} = \infty$) would shed light on the " assertion" at the beginning of this note, but it has failed to do so as yet. Senata(1967) did show that

$$E\ \tau < \infty \text{ iff } \int_0^1\ -\ \frac{1-u}{f(u)\ -u}\ du < \infty,$$

For most applications, it is of interest to know not only the probability of ultimate extinction of a family line, but also the distribution of the time to extinction. To this end, let T denote the time to extinction of the BGW branching process with probability generating function

$$g(s) = \Sigma_{j=0}^{\infty}\ p_j\ s^j,\quad 0 \leq s \leq 1\qquad \ldots\ (2.3.1)$$

where $g'(s) = m \leq 1$. In this case $P[T < \infty] = 1$. $T = n$ if and only if $Z_{n-1} > 0$ and $Z_n = 0$.

(i) If $Z_o = 1$, then

$$P[T \leq n] = P(Z_n = 0) = g_n(o),\ n \geq 1,$$

and

$$P[T = n] = P[T \leq n] - P[T \leq n-1]$$
$$= g_n(o) - g_{n-1}(o).$$

(ii) If $Z_o = k$, then

$$P[T \leq n | Z_o = k] = [P(T \leq n \mid Z_o = 1)]^k = [g_n(o)]^k$$

and

$$P[T < \infty \mid Z_o = k] = q^k = 1.$$

Some properties of the distribution of T are immediately evident. When $m < 1$, the mean value theorem and the convexity of g imply that $1-g_n(o) \leq m^n$, so that $ET^\alpha < \infty$ for some $\alpha > 0$.

Since $P[T \leq n] = g_n(o)$, the behaviour of iterates of g evaluated at 0 must be determined in order to know the distribution of T. As mentioned before, there are very few families of p.g.f.'s whose iterates have a simple closed form expressions. For other p.g.f.'s, the iterates must be calculated sequentially, and it is impractical to calculate large numbers of them. For example, the critical process form an important class in applications, because of the stationary expected size of the process; for there, $1-g_n(o) \backsim 2/[ng''(1)]$ (as pointed out in section 1.5), so the iterates evaluated at zero approach one at a slow rate. In fact there are several other branching process models for which the situation is either even worse as it is even impossible to calculate the p.g.f. for Z_n for any $n > 1$.

To circumvent this difficulty the idea of bounds or approximation for the distribution of the time to extinction for various branching models has been introduced during the last two decades; different approaches have been proposed. In most of these approaches too much attention has been focussed on one aspect at the expense of some other vital aspects. However, uniform approach taking care of the entire distribution of T has been proposed by Agresti (1974) who derived the bounds for the iterates of g and thereby obtained bounds for the parameters such as percentiles of T, ET and μ as by products of these bounds.

Seneta (1967 b) noticed that if U and L are two p.g.f.'s such that

$$L(s) \leq g(s) \leq U(s), \; 0 \leq s \leq 1$$

then, by induction

$$L_n(s) \leq g_n(s) \leq U_n(s), \; o \leq s \leq 1 \qquad \ldots \quad (2.3.2)$$

Letting s=o in (2.3.2), it follows that

$$L_n(o) \leq g_n(o) = P[T \leq n] \leq U_n(o), \; n \geq 1$$

and

$$\Sigma_{n=o}^{\infty} n^k \, [1-U_n(o)] \leq \Sigma_{n=o}^{\infty} n^k [1-g_n(o)] \leq \Sigma_{n=o}^{\infty} n^k [1-L_n(o)],$$
$$\text{for } k = 0,1,2,\ldots$$

When m<1, letting k=o, the expected time to extinction

$ET = \Sigma_{n=0}^{\infty} (1-g_n(o))$ is bounded by

$$\Sigma_{n=0}^{\infty} (1-U_n(o)) \leq ET \leq \Sigma_{n=0}^{\infty}(1-L_n(o))$$

and when in addition $g''(1) < \infty$, the mean of the Yaglom limit, $K'(1) = \mu = \lim_{n\to\infty} [m^n/(1-g_n(o))]$ is bounded by

$$\lim_{n\to\infty} \frac{m^n}{1-L_n(o)} \leq \mu \leq \lim_{n\to\infty} \frac{m^n}{1-U_n(o)}$$

2.4 Fractional Linear Generating Function Bounds of p.g.f.

The method of bounding a p.g.f. $g(s)$ by two f.l.g.f's is very useful in generating bounds for the extinction time distribution, the expected time to extinction and the percentiles γ_α of T.

Now, assuming $Z_0 = 1$, $T(m,c)$ the time to extinction of BGW branching process with f.l.g.f. $f(m, c; s)$, can be explicitly stated, putting $s=0$ in (2.2.2) we get;

(i) $(m\neq 1)$ $P[T(m,c) \leq n] = f_n(m,c;o) = \dfrac{s_o(m^n-1)}{m^n-s_o}$, or

$$P[T(m,c)=n] = f_n(m,c;o) - f_{n-1}(m,c;o) = \frac{m^{n-1}s_o(s_o-1)(1-m)}{(m^n-s_o)(m^{n-1}-s_o)},$$

$n \geq 1$

(ii) $(m=1)$ $P[T(m,c) \leq n] = nc/[1+(n-1)c]$, or

$$P[T(m,c)=n] = c(1-c)/\left[\{1+(n-1)c\}\{1+(n-2)c\}\right], \quad n \geq 1$$

Now, if $f(m,c_1;s)$ and $f(m,c_2;s)$ are two f.l.g.f.'s having the same mean m as $g(s)$, i.e.

$$f'(m,c_1;1) = f'(m,c_2;1) = g'(1) = m$$

such that

$$f(m,c_1;s) \leq g(s) \leq f(m,c_2,s), \; o \leq s \leq 1 \quad \ldots (2.4.1)$$

then, if we follow the same procedure as in section 2.3. we get

(i) $f_n(m,c_1;s) \leq g_n(s) \leq f_n(m,c_2;s), \; o \leq s \leq 1, n \geq 1 \quad \ldots (2.4.2)$

(ii) $P[T(m,c_1) \leq n] \leq P[T \leq n] \leq P[T(m,c_2) \leq n] \quad \ldots \quad (2.4.3)$

where $T(m,c_1)$ and $T(m,c_2)$ are the extinction time of the BGW branching processes with f.l.g.f.'s $f(m,c_1;s)$ and $f(m,c_2;s)$ respectively. If $m<1$,

(iii) $\Sigma_{n=0}^{\infty}(1-f_n(m,c_2;o) \leq ET \leq \Sigma_{n=0}^{\infty}(1-f_n(m,c_1;o)) \quad \ldots \quad (2.4.4)$

and if $g''(1) \leq \infty$, then

(iv) $\dfrac{m^n}{1-f_n(m,c_1;o)} \leq E(Z_n \mid Z_n > o) \leq \dfrac{m^n}{1-f_n(m,c_2;o)}$

(v) $\lim\limits_{n \to \infty} \dfrac{m^n}{1-f_n(m,c_1;o)} \leq \mu \leq \lim\limits_{n \to \infty} \dfrac{m^n}{1-f_n(m,c_2;o)} \quad \ldots (2.4.5)$

That is, fractional linear generating function bounds for $g(s)$ immediately extend to bounds for the distribution of T, expected time to extinction ET and the asymptotic conditional mean μ.

Also we can calculate the percentiles of the distribution of $T(m,c)$ explicitly. The 100 αth percentile of T is the value $Y_{\alpha}(g)$ such that $P[T \leq Y_{\alpha}(g)] = \alpha$.

Let $Y_\alpha(m,c)$ denote the 100 αth percentile of $T(m,c)$. If $m \neq 1$, n is the 100 $s_o(m^n-1)/(m^n-s_o)$ percentiles of the distribution of $T(m,c)$ in the sense that

$$P[T(m,c) \leq n] = s_o(m^n-1)/(m^n-s_o), \quad n = o,1,2,\ldots$$

If we consider $P[T(m,c) \leq n]$ as a continuous function of n for all real numbers $n \geq o$, then for $0 < \alpha < q$, $Y_\alpha(m,c)$ is the solution of

$$\text{(i)} \qquad (m \neq 1) \ \alpha = s_o[m^{y_\alpha(m,c)} - 1] / [m^{y_\alpha(m,c)} - s_o],$$

$$\text{(ii)} \qquad (m = 1) \ \alpha = C.\ Y_\alpha(m,c) / [1+c\ Y_\alpha(m,c) - 1 \ ]$$

That is, if $m \neq 1$

$$Y_\alpha(m,c) = \mathrm{Log}[(1-\alpha) / (1-\alpha/s_o)] / \log(m)$$

$$= \mathrm{Log}[(1-\alpha) / \{1-\alpha c/(1-m(1-c)\}] / \log(m) \ldots (2.4.6)$$

and if $m = 1$

$$Y_\alpha(m,c) = \alpha(1-c) / [c(1-\alpha)], \ o < \alpha < 1 \quad \ldots \qquad (2.4.7)$$

These equations express $Y_\alpha(m,c)$ as a continuous function of α. If we require $Y_\alpha(m,c)$ to be the smallest integer such that $P[T(m,c) \leq Y_\alpha(m,c)] \geq \alpha$, then $Y_\alpha(m,c)$ is the greatest integer part of (2.4.6) or (2.4.7) plus one, for $\alpha \neq f_n(m,c;o)$, $n=0,1,2,\ldots$ However, we shall use the continuous interpretation.

Now, let $Y_\alpha(m,c_1)$ and $Y_\alpha(m,c_2)$ be the 100 αth percentile of the extinction time $T(m,c_1)$ and $T(m,c_2)$ of the BGW branching processes with f.l.g.f.'s $f(m,c_1;s)$ and $f(m,c_2;s)$ respectively.

From (2.4.3)

$$P[T(m,c_1) \leq Y_\alpha(g)] \leq P[T \leq Y_\alpha(g)] = \alpha \leq P[T(m,c_2) \leq Y_\alpha(g)]$$

Then, since

$$P[T(m,c_1) \leq Y_\alpha(g)] \leq \alpha = P[T(m,c_1) \leq Y_\alpha(m,c_1)]$$

and

$$P[T(m,c_2) \leq Y_\alpha(g)] \geq \alpha = P[T(m,c_2) \leq Y_\alpha(m,c_2)]$$

We have

$$Y_\alpha(m,c_2) \leq Y_\alpha(g) \leq Y_\alpha(m,c_1), \; 0 < \alpha < q \quad \ldots \quad (2.4.8)$$

Thus, the fractional linear bounds for g directly extend to bounds for the percentiles of T.

C H A P T E R - III

CHAPTER - III

Some Asymptotic Properties of Sub-Critical Galton-Watson Process :-

3.1 Limiting Probabilities of Branching Process Whose Off-spring Distribution Depends on the Mean:-

For limiting probabilities of branching process whose off-spring distribution depends on the mean, E. Seneta in 1968 stated that, let Z_n be the number of individuals in the n^{th} generation of a discrete branching process, descended from a single ancestor, for which we put

$$F(s) = \Sigma_{j=0}^{\infty} s^j P[Z_1 = j], \quad 0 < F(o) < 1, \quad s \in [0,1]$$

It is well known that the probability generating function of Z_n is $F_n(s)$, the n-th functional iterate of $F(s)$, and that if $m = EZ_1$ does not exceed unity, then $\lim n \rightarrow \infty \; F_n(s) = 1$, $0 \leqq s \leqq 1$. In particular, extinction is certain.

For a subscritical process (i.e. $m<1$) results of Kolmogorov and Yaglom state that if $F''(1-) < \infty$

$$\lim_{n\to\infty} \frac{m^n}{1-F_n(o)} = \mu \quad , \qquad 1 \leqq \mu < \infty \qquad \ldots \qquad (3.1.1)$$

$$\lim_{n\to\infty} G_n(s) = G(s) \quad , \qquad s \in [0,1] \qquad \ldots \qquad (3.1.2)$$

exists, where

$$G_n(s) = \Sigma_{j=1}^{\infty} s^j P[Z_n = j \quad Z_n > 0] = \frac{F_n(s) - F_n(o)}{1-F_n(o)} ,$$

and G(s) is a proper generating function, with the mean of the corresponding distribution $G'(1-) = \mu$, and the corresponding variance, σ^2, finite.

In two separate papers, Heathcote and Seneta(1967) and Seneta(1967) have obtained bounds for ET, Var T and μ, where T is the time to extinction. In relation to this, he note that since

$$P[T > n] = 1 - F_n(o) \backsim \mu^{-1}.\ m^n$$

as $n \longrightarrow \infty$, all moments of thedistribution of T exist. The second of theabove mentioned papers considers only the offspring distribution, for which $F(s) = \exp m(s-1)$. Here it is shown that the bounds are sufficiently good to yield the asymptotic expressions

$$ET \backsim - \theta_m \operatorname{Log} (1-m)$$

$$\Sigma_{k=0}^{\infty} k\ P[T > k] \backsim \rho_m \frac{\pi^2}{6(1-m)}$$

$$\mu \backsim \frac{1}{\Psi_m(1-m)}$$

$$\frac{\mu^2}{\sigma^2} \backsim \frac{1}{\Psi_m - 1}$$

as $m \rightarrow 1-$, where $1 \lessapprox \Psi_m, \theta_m \lessapprox 2$. The paper conjectures also that Ψ_m, ρ_m, θ_m can in fact be replaced by the constant 2, in the above expressions. Considering a class of branching processes whose offspring distribution depends in a specific way on the mean. Senate(1968) attempts to sharpen and generalize these results dependence on mean in a specific way means that the following conditions are satisfied:

(i) $F(s) = F(m,s)$ is a p.g.f. for all m such that $1-\epsilon < m < 1$, (i.e. in some left open neighbourhood of $m=1$) and

$$F(m;s) \longrightarrow F(*;s), \text{ as } m \longrightarrow 1-, \ s \in [0,1]$$

where $F(*;s)$ is a proper p.g.f.

(ii) $F''(*;1) > 0$

(iii) $F'''(m;1) < C = \text{const.}, \ m \in (1-\epsilon, 1)$

Note: Dashes shall always refer to differentiation w.r.t.s.

By utilizing some techniques from both Heatheote and Senata(1967) and Senata(1967), together with a general approach which is basically simpler, Author shows that for this class of branching processes:

$$ET \backsim - \frac{2}{F''(*;1)} \operatorname{Log}(1-m); \quad \ldots \quad (3.1.3.)$$

$$\Sigma_{k=0}^{\infty} k^{\alpha} P[T > k] \backsim \frac{2}{F''(*;1)} \frac{\Gamma(\alpha+1)\ \zeta(\alpha+1)}{(1-m)^{\alpha}} \quad \ldots \quad (3.1.4)$$

For integral $\alpha \geqq 1$;

$$\mu \backsim \frac{F''(*;1)}{2} \cdot \frac{1}{(1-m)} \quad \ldots \quad (3.1.5)$$

$$\frac{\mu^2}{\sigma^2} \backsim 1 ; \quad \ldots \quad (3.1.6)$$

as $m \to 1-$. In this simple situation $\zeta(s)$, the Riemann zeta - function is given by

$$\zeta(\alpha+1) = \Sigma_{n=1}^{\infty} \frac{1}{n^{\alpha+1}}$$

Thus, inter alia, the conjectures for the Poision case are valid.

Since the author is not concerned with varying m, he omits explicit mention of it in the functional form. His procedure, initially that of Heathcote and Seneta(1967) viz. using first and second order mean-value theorems, differs from Heathcote and Seneta(1967) in that he obtains bounds not for $\{1-F_n(o)\}$, but rather its reciprocal.

Sinc $F_{k+n}(s) = F_n[F_k(s)]$, for integral n, $k \geq 0$

$$1-F_{k+1}(o) = \{1-F_n(o)\} F'(\theta_k)$$

where $F_k(o) < \theta_k < 1$, we have by monotonoeity of $F'(s)$ < (and since $F_k(o) \uparrow 1$ as $k \to \infty$) that for $0 \leq h \leq k$

$$F'(F_h(o)) \{1-F_k(o)\} \leq 1-F_{k+1}(o) \leq m\{1-F_k(o)\} \quad \ldots(3.1.7)$$

Moreover, for $k \geq 0$

$$F_{k+1}(o) = F(F_k(o)) = 1-\{1-F_k(o)\} m + \frac{F''(\eta_k)}{2} . \{1-F_k(o)\}^2$$

where $F_k(o) < \eta_k < 1$, so that putting $b_k = \{1-F_k(o)\}^{-1}$ we have

$$b_{k+1} = \frac{b_k}{m} + \frac{1}{m} . \frac{F''(\eta_k)}{2} . \frac{b_{k+1}}{b_k} \ldots \quad (3.1.8)$$

Now, since $F''(\eta_k) \geqq F''(F_k(o)) \geqq F''(F_h(o))$ for $o \leqq h \leqq k$, and using (3.1.7)

$$\frac{b_k}{m} + \frac{1}{m} \cdot \frac{F''(F_h(o))}{2m} \leqq b_{k+1} \leqq \frac{b_k}{m} + \frac{1}{m} \cdot \frac{F''(1)}{2} \cdot \frac{1}{(F'(F_h(o))} \quad ..(3.1.9)$$

(assume that $0<F''(1) < \infty$). This inequality is the crucial one from which all subsequent results follow. Keeping h fixed and interating,

$$\frac{b_h}{m^n} + \frac{F''(F_h(o))}{2m^2} \cdot \Sigma_{i=0}^{n-1} \frac{1}{m^i} \leqq b_{h+n} \leqq \frac{b_h}{m^n} + \frac{F''(1)}{2mF'(F_h(o))} \cdot \Sigma_{i=o}^{n-1} \frac{1}{m^i}$$

We then have

$$m^h b_h + \frac{F''(F_h(o)).m^h(1-m^n)}{2m(1-m)} \leqq m^{h+n}.\ b_{h+n} \leqq m^h b_h + \frac{F''(1)m^h(1-m^n)}{2F'(F_h(o))(1-m)} \quad .... (*)$$

and letting $n \to \infty$

$$m^h b_h(1-m) + \frac{F''(F_h(o))m^h}{2m} \leqq (1-m)\mu \leqq m^h b_h(1-m) + \frac{F''(1).m^h}{2F'(F_h(o))} \quad ... (3.1.10)$$

Also from (*) above,

$$\frac{2F'(F_h(o))(1-m)}{F''(1)} \left\{\frac{\theta m^n}{1-\theta m^n}\right\} \leqq \frac{1}{b_{h+n}} \leqq \frac{2m(1-m)}{F''(F_h(o))} \cdot \left\{\frac{\tau m^n}{1-\tau m^n}\right\}$$

where

$$1 > \theta = \frac{F''(1)}{2b_h F'(F_h(o))(1-m)+F''(1)} > 0;$$

$$1 > \tau = \frac{F''(Fh(o))}{2m(1-m)b_h+F''(F_h(o))} > 0$$

so that for integral $\alpha \geqq 0$

$$\Sigma_{k=o}^{h} k^{\alpha} \{ 1- F_k(o) \} + \frac{2F'(F_h(o))(1-m)}{F''(1)} \cdot \Sigma_{n=1}^{\infty} (h+n)^{\alpha} . \left\{ \frac{\theta m^n}{1-\theta m^n} \right\}$$

$$\leqq \Sigma_{k=o}^{\infty} k^{\alpha} P[T > k] \qquad \ldots \qquad (3.1.11)$$

$$\leqq \Sigma_{k=o}^{h} k^{\alpha} \{ 1-F_k(o) \} + \frac{2m(1-m)}{F''(F_h(o))} \cdot \Sigma_{n=1}^{\infty} (h+n)^{\alpha} \left\{ \frac{\gamma m^n}{1-\gamma m^n} \right\}.$$

The two sets of inequalities (3.1.10) and (3.1.11) shall be sufficient to give him the required asymplotic results by suitable limiting consideration. He begins with some remarks on the sums occuring in the bounds (3.1.11), for $\alpha=0,1$. Since from Senata(1967) for $0 < \rho, s < 1$

$$\frac{\log(1- \rho s)}{\log s} \leqq \Sigma_{j=1}^{\infty} \frac{s^j}{1- \rho s^j} \leqq \frac{\rho s}{1-\rho s} + \frac{\log(1- \rho s)}{\log s}$$

if $\rho(=\rho(s))$ is such that, as $s \longrightarrow 1-$

$$(1- \rho s) \backsim C.(1-s) \qquad (0<c = \text{const.})$$

it follows that

$$\lim_{s \to 1-} \left\{- \frac{(1-s)}{\log(1-s)} \cdot \Sigma_{j=1}^{\infty} \frac{\rho s^j}{1- \rho s^j} \right\} = 1 \qquad \ldots (3.1.12)$$

Moreover, it was shown in Senata(1967), that

$$\lim_{s \to 1-} \left\{ \frac{6(\log s)^2}{\pi^2} \cdot \Sigma_{j=1}^{\infty} \frac{j \rho s^j}{1- \rho s^j} \right\} = 1 \quad \ldots (3.1.13)$$

providing $1- \rho k.(1-s)$ as $s \longrightarrow 1-$ where $o<k = \text{const.}$

Since the procedure for sums of the form,

$$\Sigma_{j=1}^{\infty} \frac{j^{\alpha} \rho s^j}{1- \rho s^j}$$

is a slight extension, essentially, of the procedure to obtain

(3.1.13), in the cited reference, he only outline it here. The remarks apply for integral $\alpha \geqq 1$, $0<\rho$, $s<1$.

(a) The function of a continuous variable $x>0$

$$\frac{x^{\alpha}\rho s^{x}}{1-\rho s^{x}}$$

has a unique maximum at $x=N^*$ which is the unique solution of

$$\alpha + x\log s - \alpha\rho s^{x} = 0.$$

Hence N^* satisfies

$$0 \leqq \alpha\rho s^{N^*} = \alpha + N^* \log s \Longrightarrow N^* \leqq -\frac{\alpha}{\log s},$$

$$0 = \alpha\left(1+\frac{N^*}{\alpha}\log s - \rho s^{N^*}\right) \geqq \alpha\left(1+N^*\log s - \rho s^{N^*}\right) \Longrightarrow N^* \gtrsim \frac{\sqrt{2k}}{\sqrt{-\log s}}$$

if $1-\rho \sim k.(1-s)$, as $s \longrightarrow 1-$, where $k>0$ is independent of s. To see the validity of last asymptotic inequality, he has

$$0 \geqq 1 + N^* \log s - \rho e^{-(-N^*\log ss)}$$

and since for $x \geqq 0$,

$$e^{-x} \leqq 1-x+\frac{x^2}{2}$$

$$0 \geqq 1+N^*\log s - \rho\left\{1+N^*\log s + \frac{(N^*\log s)^2}{2}\right\},$$

i.e.

$$0 \geqq (1-\rho)+(1-\rho)N^*\log s - \rho.\frac{(N^*\log s)^2}{2}$$

This is just a quadratic inequality for $N^*>0$, whose solution, as $s \longrightarrow 1-$, (if $1-\rho \sim k.(1-s)$) is given by

$$N^* \gtrsim \frac{\sqrt{2k}}{\sqrt{-\log s}}$$

Hence, if $1-\rho \sim k.(1-s)$, we have

$$\frac{(N^*)^\alpha \rho s^{N^*}}{1-\rho s^{N^*}} \lesssim \frac{\text{Const.}}{(-\log s)^{\alpha+(1/2)}} \qquad (0 < \text{const.} < \infty)$$

as $s \longrightarrow 1-$

(b) From a double use of the Caueby integral

$$\Sigma_{j=0}^{\infty} \frac{j^\alpha \rho s^j}{1-\rho s^j} = \int_0^\infty \frac{x^\alpha \rho s^x}{1-\rho s^x} \cdot dx + \varepsilon(s)$$

where

$$|\varepsilon(s)| \leqq \text{const.} \frac{(N^*)^\alpha \rho s^{N^*}}{1-\rho s^{N^*}} .$$

(c) $$\int_0^\infty \frac{x^\alpha \rho s^x}{1-\rho s^x} \cdot dx = -\frac{1}{(\log s)^{\alpha+1}} \int_0^\rho \frac{(\log y - \log \rho)^x}{1-y} dy.$$

(d) $$\int_0^1 \frac{(\log y)^\alpha}{1-y} \cdot dy = (-1)^\alpha / \overline{(\alpha+1)} \ \zeta(\alpha+1),$$

for $\alpha \geqq 0$ integral

An obvious combination of these results shows that for $\alpha \geqq 1$ and integral, if $1-\rho \sim k.(1-s)$ as $s \rightarrow 1-$,

$$\Sigma_{j=1}^{\infty} \frac{j^\alpha \rho s^j}{1-\rho s^j} \ \frac{\Gamma(\alpha+1)\, \zeta(\alpha+1)}{(-\log s)^{\alpha+1}} \qquad \ldots (3.1.14)$$

Secondly he remarks that under his condition (i), (ii) and (iii) of branching processes whose offspring distribution depend on the mean, as $m \longrightarrow 1-$

(a') $F_k(m;s) \longrightarrow F_k(*;s)$, $\quad s \in [0,1]$

(b') $F'(m;F(m;0)) \longrightarrow F'(*;F_h(*;0))$

(c') $F''(m;F_h(m;0)) \longrightarrow F''(*;F_h(*;0))$

(d') $F'(*;1) = 1$

(e') $F''(m;1) \longrightarrow F''(*;1)$, $0 < F''(*;1) < \infty$

To prove (a') consider the inequality

$$\left|F_k(m;s) - F_k(*;s)\right| \leqq F_{k-1}(m;F(m;s)) - F_{k-1}(m;F(*;s)) + F_{k-1}(m;F(*;s)) - F_{k-1}(*;F(*;s))$$

and notice that the first part tends to zero by the mean value theorem.

$$\left| F_{k-1}(m;F(m;s)) - F_{k-1}(m;F(*;s))\right| = F'_{k-1}(m;\delta_m) \left| F(m;s)-F(*;s) \right|$$

where $0 \leqq \delta_m \leqq 1$ so that $F'_{k-1}(m;\delta_m)(\leqq F'_{k-1}(m;1) = m^{k-1})$ is bonded as $m \rightarrow 1-$, and $F(m;s) \longrightarrow F(*;s)$ by (i). The second part of the right hand side approaches zero by induction on k, and (i). Propositions (b') and (c') are proved by analogous arguments: consider (c'):

$$\left| F''(m;F_h(m;0))-F''(*;F_h(*;o))\right| \leqq \left|F''(m;F_h(m;o) - F''(m;F_h(*;o))\right| + \left|F''(m;F_h(*;o)-F''(m;F_h(*;o))\right|$$

Here let him focus attention first on

$$\left| F''(m;F_h(*;o)) - F''(*;F_h(*;o)) \right|$$

in which he notice that

$$j(j-1)P[Z_1=j][F_h(*;o)]^j \leqq j(j-1)[F_h(+;o)]^j,$$

where the right hand side is independent of m, and so

$$F''(m;F_h(*;o)) = \Sigma_{j=0}^{\infty} j(j-1)P[Z_1=j][F_h(*;o)]^{j-2} \leqq \Sigma_{j=0}^{\infty} j(j-1)[F_h(*;o)]^{j-2} = 2[1-F_h(*;o)]^{-3} < \infty$$

since $o < F_h(*;o) < 1$. Thus by dominated convergence of the series for $F''(m;F_h(*;o)$ and since the assumption (i) implies coefficient convergence in $F(m;s)$ to $F(*;s)$, it follows that as $m \longrightarrow 1-$

$$\left| F''(m;F_h(*;o)) - F''(*;F_h(*;o)) \right| \longrightarrow o .$$

On the other hand,

$$\left| F''(m;F_h(m;o) - F''(m;F_h(*;o) \right| = F'''(m;\theta_m) \left| F_h(m;o) - F_h(*;o) \right|$$

where $o \leqq \theta_m \leqq 1$; and so as $m \longrightarrow 1-$ (since $F'''(m;1)$ is bounded, by (iii) and from (a') above) we get the requisite tendency to zero.

Propositions (d') and (e') follow since condition (i) implies convergence in distribution as $m \longrightarrow 1-$, and condition (iii) is equivalent to uniform boundness of the third moment as $m \longrightarrow 1-$. Hence by a well-known corollory of the moment convergence theorem, he has convergence of the first and second moments to those of the limit distribution, which are necessarily finite. Condition (ii) completes assertion (e').

In concluding, he note that

$$o < F(*;o) < 1$$

this being implied by (d') and (e') and since from (d') also $F(*;1) = 1$, the branching process defined by $F(*;s)$ is critical, and extinction is therefore certain, i.e.

$$F_h(*;o) \uparrow 1 \quad \text{as } h \longrightarrow \infty$$

After having discussed the preliminaries the author combines these results to deduce first (3.1.3) and (3.1.4) and then (3.1.5) and (3.1.6).

To obtain (3.1.3), consider (3.1.5), with $\alpha=0$ and identify θ and τ successively with $\varphi = \varphi(s)$ and s with m. (Note also, that since $F''(m;1) \longrightarrow F''(*;1)$ as $m \to 1-$, $\infty > F''(m;1) > 0$ for m sufficiently close to unity). First notice that he has putting $\varphi = \theta$, $s=m$

$$1-\varphi s = 1-\theta m = (1-m)\,\frac{2\{1-F_h(m;o)\}^{-1}\,F'(m;F_h(m;o)) + F''(m;1)}{2\{1-F_h(m;o)\}^{-1}\,F'(m;F_h(m;o))(1-m)+F''(m;1)}$$

$$\sim C.(1-m) \qquad (o < C \equiv C(h) < \infty) \quad \ldots \quad (3.1.15)$$

as $m \to 1-$. This result, which amounts to saying that the part in square brackets approaches $C \equiv C(h)$ as $m \to 1-$, is a direct consequence of the above propositions and above conditions (i) - (iii). So also

$$1-\tau m \quad K.(1-m) \qquad (o < k \equiv k(h) < \infty) \quad \ldots \quad (3.1.16)$$

as $m \longrightarrow 1-$

The remarks (3.1.15) and (3.1.16) make it possible to apply (3.1.12) to the bounds of (3.1.11) with $\alpha = 0$, which are of the form required by (3.1.12) after division throughout by- $\log(1-m)$. Letting $m \to 1-$, we obtain, therefore using also the results of (a'), (b'), (c'), d') and (e'),

$$\frac{2F'(*;F_h(*;o))}{F''(*;1)} \leqq \lim_{m\to 1-} \inf.\left\{\frac{-ET}{\log(1-m)}\right\}$$

$$\leqq \lim_{m \to 1-} \sup \left\{ \frac{-ET}{\log(1-m)} \right\}$$

$$\leqq \frac{2}{F''(*;F_h(*;o))}$$

Note: Until this point, h has been fixed, but arbitrary.

Now, $F(*;s)$ is a proper generating function with $F'(*;1) = 1$, $F''(*;1) > 0$. Hence from the well known extinction property in this case, as pointed out $F_h(*;o) \uparrow 1$ as $h \to \infty$. Since, in the above expression, h may be made arbitrarily large, and he has the necessary dominated convergence,

$$\lim_{m \to 1-} \left\{ \frac{-ET}{\log(1-m)} \right\} = \frac{2}{F''(*;1)}$$

which is (3.1.3) as required.

To prove (3.1.4), he consider (3.1.11) with integral $\alpha \geqq 1$; he observed more detail than in case $\alpha = 0$, since the situation is slightly more complex. Consider the right hand inequality of (3.1.11):

$$\Sigma_{k=o}^{\infty} k^{\alpha} P[T > k]$$

$$\leqq \Sigma_{k=o}^{\infty} k^{\alpha} \left\{ 1-F_h(o) \right\} + \frac{2m(1-m)}{F''(F_h(o))} \cdot \Sigma_{n=1}^{\infty} (h+n)^{\alpha} \left\{ \frac{\Upsilon m^n}{1-\Upsilon m^n} \right\}$$

.... (3.1.17)

where h is arbitrary and fixed. He notice first that $1-\Upsilon \sim k' \cdot (1-m)$ as $m \to 1-$ where $o < k' = k'(h) < \infty$. Identifying Υ and m with Υ and s then have from (3.1.14) (since $\alpha \geqq 1$) that as $m \to 1-$

$$\Sigma_{j=1}^{\infty} \frac{j^{\alpha}\gamma_m{}^j}{1-\gamma_m{}^j} \backsim \frac{\Gamma(\alpha+1)\,\zeta(\alpha+1)}{(-\log m)^{\alpha+1}} .$$

Thus multiplying (3.1.17) by $(1-m)^{\alpha}/\Gamma(\alpha+1)\,\zeta(\alpha+1)$ he has as $m \rightarrow 1-$.

$$\lim_{m\rightarrow 1-} \sup\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\,\zeta(\alpha+1)} \cdot \Sigma_{k=o}^{\infty} k^{\alpha} P[T>k]\right\}$$

$$\leqq \lim_{m\rightarrow 1-} \left\{\frac{2m(1-m)^{\alpha+1}}{\Gamma(\alpha+1)\,\zeta(\alpha+1)F''(m;F_h(m;o))} \cdot \Sigma_{n=1}^{\infty} n^{\alpha}\left\{\frac{\gamma_m{}^n}{1-\gamma_m{}^n}\right\}\right\}$$

$$= \frac{2}{F''(*;F_h(*;o))}$$

since the remaining contributing terms of the right hand side of (3.1.17) are $O\left\{-(1-m)\log(1-m)\right\}$ as $m \rightarrow 1-$.

The left hand inequality of (3.1.11) may be treated in the same way, since $1-\theta \backsim C'.(1-m)$ $(0< C' = C'(h) < \infty)$ as $m \rightarrow 1-$, so that he get eventually

$$\frac{2F'(*;F_h(*;o))}{F''(*;1)} \leqq \lim_{m\rightarrow 1-} \inf.\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\,\zeta(\alpha+1)} \cdot \Sigma_{k=o}^{\infty} k^{\alpha}P[T > k]\right\}$$

$$\leqq \lim_{m\rightarrow 1-} \sup.\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\,\zeta(\alpha+1)} \cdot \Sigma_{k=o}^{\infty} k^{\alpha}P[T > k]\right\}$$

$$\leqq \frac{2}{F''(*;F_h(*;o)}$$

Thus once more letting $h \rightarrow \infty$, he get the required result(3.1.4).

$$\lim_{m\rightarrow 1-}\left\{\frac{(1-m)^{\alpha}}{\Gamma(\alpha+1)\,\zeta(\alpha+1)} \cdot \Sigma_{k=o}^{\infty} k^{\alpha}P[T > k]\right\} = \frac{2}{F''(*;1)} .$$

To obtain (3.1.5) and (3.1.6) he return to (3.1.10) where first letting $m \to 1-$, and then letting $h \to \infty$ yields

$$\lim_{m\to 1-} (1-m)\mu = \frac{F''(*;1)}{2}$$

Since,

$$\sigma^2 = \frac{v^2\mu}{m(1-m)} - \mu^2$$

where

$$v^2 = \text{Var } Z_1 = F''(m;1) + F'(m;1) - \{F'(m;1)\}^2$$

it is easily shown that

$$\lim_{m\to 1-} \frac{\mu^2}{\sigma^2} = 1$$

as required.

Some remarks on the class of distributions defined by conditions (i), (ii) and (iii) are in order. The condition (i) is one which renders the procedure $m \to 1-$ meaningful; (iii) ensures the convergence of first and second moments and is also used to prove assertion (C'). Neither of these is open to obvious relaxation, as their role is relatively clear cut.

On the other hand condition (ii) is obviously necessary to give the correct asymptotic behaviour in formulae (3.1.3) - (3.1.6), and a relaxation of this condition is of interest in that he is concerned as to how this changes the behaviour as $m \to 1-$. First he notice that $F''(*;1) = 0$, in view of (i) and (d') implies $F(*;s) = s$; in fact, as pointed out, $F''(*;o) > 0$ renders $F(*;s)$ a sensible p.g.f. for a branching process, and since $F'(*;1) = 1$, enables him to say $F_h(*;o) \to 1$ as $h \to \infty$, a most important step in his arguments.

Nevertheless, when $F''(*;1) = 0$, some deductions are possible, if he makes some further assertion. He shall only consider one such in general viz. $F''(m;1) > 0$ for all m sufficiently close to one. A careful consideration of the bounds reveals that in this case (as expected from (3.1.3) and (3.1.5))

$$\lim_{m \to 1-} (1-m)\mu = 0 \qquad \ldots \qquad (3.1.18)$$

$$\lim_{m \to 1-} \left\{ \frac{ET}{-\log(1-m)} \right\} = \infty . \qquad \ldots \qquad (3.1.19)$$

An example of such a distribution is given by the probability generating function of bilinear fractional form

$$F(m;s) = 1-m^2 + \frac{m^3 s}{1-(1-m)s}, \quad 0 < m < 1$$

which defines a modified grometric distribution. In this case we can calculate μ and σ^2 and obtain ET asymptotically:

$$\mu = \frac{1+m}{m} \longrightarrow 2$$

$$\sigma^2 = \frac{(1+m)}{m^2} \longrightarrow 2$$

$$ET \backsim \frac{\log 2}{(1-m)}$$

as $m \to 1-$, which agrees with (3.1.18) and (3.1.19). Note also $\mu^2/\sigma^2 \longrightarrow 2$ as $m \to 1-$.

The extremely pathological case not covered by any of the above is the two-point offspring distribution

$$F(m;s) = (1-m) + ms, \quad 0 < m < 1$$

since $F''(m;1) = 0$ all $m \in (0,1)$. In this case

$$\frac{m^k}{1 - F_k(m;o)} = 1$$

for all $k \geqq 0$ and

$$ET = \Sigma_{k=o}^{\infty} \left\{1 - F_k(m;o)\right\} = \Sigma_{k=o}^{\infty} m^k = [1-m]^{-1}$$

which seems to behave analogously to the case just discussed.

In conclusion, he pointed out the relation of some of the present results, to relevent ones in the literature.

It was pointed out in Senata(1967) that a diffusion approximation result of Feller, as " $m \to 1-$ " suggested the validity of (3.1.5) and (3.1.6) in a wide class of cases for which '$m \to 1-$' had a meaning. Another result of more immediate relevence in relation to this is the apparent assertion of Nagaev and Muhamedhanova(1966) that if he put for his branching process (under conditions closely resembling (i),(ii) and (iii))

$$S_n(y) = P\left[\left\{Z_n\left(\frac{1-F_n(o)}{m^n}\right)\right\} < y \,\middle|\, Z_n > 0\right]$$

then

$$S_n(y) \to \begin{cases} 1-e^{-y} & y \geqq o \\ o & y < o \end{cases}$$

as $n \to \infty$, and $m \to 1+$ or $m \to 1-$, which certainly suggests that, as $m \to 1-$

$$\frac{\sigma^2}{\mu^2} \to 1,$$

from considerations (3.1.1) and (3.1.2).

Finally, the procedure in his main discussion via inequality (3.1.9), was suggested to the author by the proof of Lemma 1 in Nagaev and Muhamedhanova(1966) where the expression (3.1.8) occurs. There is no other overlap in actual content: in fact the proofs of Nagaev and Muhamedhanova seem to concentrate equally on the case $m > 1$ and thus do not consider time to extinction at all.

3.2 Offspring Distribution and the Limiting Distribution of the Population Size Conditioned on Non-Extinction:-

Bagley(1982) in his paper titled, " Asymptotic Properties of Subcritical Galton-Watson Processes" , has obtained for a supercritical GW process. Some interesting results connecting the distribution of the offspring random variable with the limiting distribution of the population size conditioned on non-extinction. The convergence of moments, both integral and non-integral, of these distributions, is also investigated. Results similar to these for supercritical case have been obtained by Bingham and Doney(1974) and similar methods have been used by De Meyer and Teugels(1980), to obtain results in quewing theory.

Let $\{Z_n\}$ denote a Galton-Watson process with $Z_o=1$ and Laplace transform $f(s) = E[\exp(-sZ_1)]$. Denote the mean offspring size $E[Z_1]$ by m, where $0<m<1$. Let Y_n be a random variable with the same distribution as Z_n conditioned on non-extinction. Let $g(s) = \lim_{n\to\infty} E[\exp(-sY_n)]$, and denote $-g'(0+)$ by μ, wherever it exists finitely.

Yoglom(1947) showed that $g(s)$ exists and that $g(s)$ and $f(s)$ satisfy the functional equation $1-g(-\log f(s)) = m(1-g(s))$. The original proof assumed the finiteness of the second derivative of $f(s)$ at $s=1$, but a sharper statement of Yaglom's theorem is given below.

<u>Theorem(3.2.1)</u>:-

If $\{Z_n\}$ is a subcritical Galton-Watson process then the following statements are equivalent:

(i) $\lim_{n\to\infty} P(Z_n=k \mid Z_n > 0) = b_k$ exists $\forall k \in N$

where $\Sigma_{k=1}^{\infty} = 1$ and $\mu = \Sigma_{k=1}^{\infty} kb_k < \infty$.

(ii) $E[Z_1 \log Z_1] < \infty$

Furthermore, if $g(s) = \Sigma_{k=1}^{\infty} b_k \exp(-sk)$ then

$$1-g(-\log f(s)) = m(1-g(s)) \quad \ldots \quad (3.2.1)$$

Using(3.2.1), Evans(1978) showed that if $f''(1-)$ exists $\mu \leqq (\mathrm{Var}\ Z_1)(m-m^2)^{-1}$ with equality if and only if $f(s) = 1-m+ms$.

Using Laplace-Stieltjes transform results given in Bingham and Doney(1974), and the basic equation(3.2.1). It is assumed throughout the remainder of the paper that $E[Z_1 \log Z_1]$ is finite, which by Theorem (3.2.1) implies the existence of μ.

Let Y be the random variable with moment generating function $g(s)$.

Theorem(3.2.2):-

For $n=2,3,\ldots,E[Z_1{}^n] < \infty$ if and only if $E[Y^n] < \infty$.

Let $L(.)$ denote a measurable, slowly varying function defined on $(0, \infty)$.

Theorem(3.2.3) :-

If $\alpha > 1$ is not an integer then

$$P(Z_1 > x) \backsim L(x)/x^\alpha \qquad (x \to \infty)$$

if and only if $P(Y > x) \backsim \mu L(x)/x^\alpha\ (m-m^\alpha)(x \to \infty)$.

Corollary(3.2.1) :- For every $\beta > \alpha$,

$$E[Z_1{}^\beta\ I_{\{0 \leqq Z_1 \leqq x\}}] \backsim \frac{\beta x^{\beta-\alpha}\ L(x)}{\beta - \alpha} \qquad (x \to \infty).$$

if and only if

$$E[Y^\beta I_{\{0 \leqq y \leqq x\}}] \backsim \frac{\beta x^{\beta-\alpha}\ L(x)\mu}{(\beta-\alpha)(m-m^\alpha)} \qquad (x \to \infty).$$

Theorem (3.2.4)

If $n=2,3,\ldots,$ and L tends to ∞ at ∞, then

$$E[Z_1^n\ I_{\{0 \leqq Z_1 \leqq x\}}] \backsim L(x) \qquad (x \to \infty)$$

if and only if $E[YI_{\{0 \leqq y \leqq x\}}] \backsim \mu L(x)/(m-m^n) \qquad (x \to \infty)$.

Theorem(3.2.5)

If $n=2,3,\ldots,$ and $E[Z_1^n] < \infty$, then

$$E[Z_1^n\ I_{\{Z_1 > x\}}] \backsim L(x) \qquad (x \to \infty)$$

if and only if $E[Y^n I_{\{y > x\}}] \backsim \mu L(x)/(m-m^n) \qquad (x \to \infty)$.

Theorem(3.2.6)

If $\int_1^\infty x^{-1}L(x)\,dx$ converges, then

$$E[Z_1 I_{\{Z_1>x\}}] \sim L(x) \qquad (x \to \infty)$$

implies

$$E[Y\, I_{\{y>x\}}] \sim \frac{-\mu}{m\log m}\int_x^\infty t^{-1}(t)\,dt \quad (x \to \infty).$$

Theorem(3.2.7)

If $\alpha > 1$ is not an integer then

$E[Z_1^\alpha L(Z_1)] < \infty$ if and only if $E[Y^\alpha L(y)] < \infty$.

Taking L to be constant, the following corollary is obtained.

Corollary(3.2.2) :- If $\alpha > 1$, $E[Z_1^\alpha] < \infty$ if and only if $E[Y^\alpha] < \infty$. Let L_0 be a positive function slowly varying at ∞.

Theorem(3.2.8) :-

(i) Let L be such that $L(x) = \int_1^x t^{-1}L_0(t)\,dt \quad (x>1)$.
If $n=2,3,\ldots$, then
$E[Z_1^n L(Z_1)] < \infty$ if and only if $E[Y^n L(y)] < \infty$.

(ii) Let $L(x)=\int_x^\infty t^{-1}L_0(t)\,dt$ $(x>1)$. If $\int_{-1}^\infty t^{-1}L_0(t)\,dt$ converges, then for $n=1,2,\ldots$,

$E[Z_1^{n+1}L(Z_1)] < \infty$ if and only if $E[Y^{n+1}L(y)] < \infty$.

Recall that Z_1 and W are connected by the functional equation

$$1-g(-\log f(s)) = m(1-g(s)) \qquad \ldots (3.2.2)$$

Let $t = -\log f(s)$ and $(s) = m\, g(s) - 1 + \mu s\, /s$.

<u>Proposition(3.2.1)</u> :- For $n=2,3,\ldots\ldots$, there exist constants $m_2, \ldots\ldots, m_n$ such that

$$f_n(s) = (-1)^{n+1}\left\{f(s) - 1 + ms - \Sigma_{r=2}^{n} m_r(-s)^r/r!\right\} = O(s^n)\,(s \to o) \quad \ldots \quad (3.2.3)$$

if and only if there exist constants $\mu_2, \ldots, \mu_n$ such that

$$g_n(s) = (-1)^{n+1}\left\{g(s) - 1 + \mu s - \Sigma_{r=2}^{n} \mu_r(-s)^r/r!\right\} = O(s^n)\,(s \to 0) \quad \ldots \quad (3.2.4)$$

<u>Proof</u>:- By induction. First assume that $E(Z_1^r) = m_r < \infty$ for $1 \leqq r \leqq n+1$, and that $\mu_r = (-1)^r\, g^{(r)}(0+) < \infty$ for $1 \leqq r \leqq n$. By the remark preceeding Theorem A of Bingham, N.H. and Doney, R.A.(1974).

$$f(s) = 1 - ms + \ldots + (-1)^n\, m_{n+1}\, s^{n+1}/(n+1)! + o(s^{n+1})$$

so that

$$t = ms + a_2 s^2 + \ldots\ldots + a_{n+1}s^{n+1} + o(s^{n+1}) \quad \ldots \quad (3.2.5)$$

where $a_2, \ldots, a_{n+1}$ are constants. Also

$$g(s) = 1 - \mu s + \ldots + (-1)^n\, \mu_n s^n/n! + (-1)^{n+1}\, g_n(s) \ldots (3.2.6)$$

and

$$\begin{aligned}\Psi(s) - \Psi(ms) &= m(g(s) - 1 + \mu s)s^{-1} - (g(ms) - 1 + \mu ms)s^{-1} \\ &= s^{-1}(g(t) - g(ms)) \qquad \ldots\ldots \quad (3.2.7)\end{aligned}$$

Combining (3.2.5), (3.2.6) and (3.2.7) we find that there exist constants $b_1, b_2, \ldots, b_n$ such that

$$\begin{aligned}\Psi(s) - \Psi(ms) = s^{-1}\{&b_1 s^2 + b_2 s^3 + \ldots + b_n\, s^{n+1} \\ &+ (-1)^{n+1}(g_n(t) - g_n(ms) + o(s^{n+1}))\} \quad \ldots\ldots \quad (3.2.8)\end{aligned}$$

To complete the proof, the following lemma is needed.

<u>Lemma (3.2.1)</u> :-

For $n \geqq 2$, if $m_2 < \infty$ and $\mu_n < \infty$,

Here $g_n(t) - g_n(ms) = o(s^{n+1})$.

<u>Proof</u> :- By (3.2.5), $t = ms + k(s)$ where $k(s)/s \rightarrow o$ and $k(s)/s^2 \rightarrow a_2$ as $s \rightarrow o$, so that

$$\{g_n(t) - g_n(ms)\}/s^{n+1} = [\{g_n(t) - g_n(ms)\}/s^{n-1}(t-ms)] \; k(s)/s^2.$$

Therefore it suffices to show that

$$\{g_n(ms+k(s)) - g_n(ms)\}/k(s)s^{n-1} \rightarrow 0 \text{ as } s \rightarrow o \quad \ldots(3.2.9)$$

By the hypothesis of lemma, $g_n'(x)/x^{n-1} \rightarrow o$ as $x \rightarrow o$, so by the mean value theorem and (3.2.9),

$$\{g_n(ms+k(s)) - g_n(ms)\}/k(s)s^{n-1} = g_n'(ms+\mathcal{E}_s)/s^{n-1}$$

where $0 \leqq |\mathcal{E}_s| \leqq |k(s)|$ and

$$g_n'(ms+\mathcal{E}_s)/m^{n-1}s^{n-1} \backsim g_n'(ms+\mathcal{E}_s)/(ms+\mathcal{E}_s)^{n-1}$$

which tends to 0 as $s \rightarrow 0$. Hence the lemma is proved.

Therefore by lemma(3.2.1) and (3.2.8)

$$\Psi(s) - \Psi(ms) = \Sigma_{r=1}^{n} b_r s^r + o(s^n).$$

Substituting s by ms, ms^2, ..., and adding he find that

$$\Psi(s) = \Sigma_{r=1}^{n} b_r s^r/(1-m^r) + o(s^n)$$

which in turn implies that $g_{n+1}(s) = o(s^{n+1})$.

Conversely, assume that (3.2.4) holds, so that

$$mg_n(s) - g_n(t) = (-1)^{n+1}\ m\left\{g(s)-1 + \mu s-\Sigma_{r=2}^{n}\mu_r(-s)^r/r!\right\}$$

$$- (-1)^{n+1}\left\{ g(t)-1+\mu t- \Sigma_{r=2}^{n}\mu_r(-t)^r/r! \right\}$$

$$= (-1)^{n+1}\left\{mg(s)-m+m\mu s- \Sigma_{r=2}^{n}m\mu_r(-s)^r/r! \right.$$
$$\left. -(g(t)-1)-\mu t+\Sigma_{r=2}^{n} \mu_r(-t)^r/r! \right\}$$

which using (3.2.2), is equal to

$$(-1)^{n+1}\left\{\mu e^{-t}-\mu e^{-t}+m\mu s-\mu t-\Sigma_{r=2}^{n} \mu_r m(-s)^r/r! +\Sigma_{r=2}^{n} \mu_r(-t)^r/r!\right\} \quad \ldots. \quad (3.2.10)$$

Also from (3.2.2) we have

$$\mu t - \mu_2 t^2/2! + \ldots.+ (-1)^n\mu_n t^n/n! + o(t^n)$$

$$= m(\mu s-\mu_2 s^2/2! + \ldots..+(-1)^n\mu_n s^n/n!) + o(s^n)$$

which implies that $t= ms + d_2s^2+ \ldots..+ d_ns^n + o(s^n)$

where $d_2, \ldots., d_n$ are constants. So substituting in (3.2.10), we find that

$$\mu^{-1}(mg_n(s) - g_n(t)) = f_n(s) + O(t^{n+1}). \quad \ldots. \quad (3.2.11)$$

By hypothesis the left hand side is $o(s^n)$, therefore $f_n(s)=o(s^n)$. Hence the proposition is proved.

<u>Corollary(3.2.3)</u> :- For $n \geqq 2$, if $f_n(s) = o(s^n)$ or

$$g_n(s) = o(s^n),$$

$$mg_n(s) - g_n(ms) = \mu f_n(s) + O(t^{n+1}) \quad \ldots. \quad (3.2.12)$$

<u>Lemma(3.2.2)</u>:- If either $f_1(s) - h(s)$ or $g_1(s) \backsim \mu m^{\alpha+1}$ $(m-m^{\alpha})\ h(s)$ holds, then $mg_1(s) - g_1(ms) = \mu f_1(s) + o(h(s))$.

<u>Proposition (3.2.2)</u>: If $n=1,2, \ldots,$ and $n < \alpha < n+1$, then

$$f_n(s) \backsim s^{\alpha} L(1/s) \quad (s \to o)$$

if and only if $g_n(s) \backsim \mu s^{\alpha} L(1/s)(m-m^{\alpha})^{-1}$ $(s \to o)$.

<u>Proposition(3.2.3)</u>: If $n=1,2, \ldots,$ and $L \to \infty$ at ∞ then

$$f_n(s) \backsim s^{n+1}\ L(y_s) \quad (s \to o)$$

if and only if $g_n(s) \backsim \mu s^{n+1}\ L(1/s)(m-m^{\alpha})$ $(s \to o)$.

<u>Proposition (3.2.4)</u>:- If $n=2,3, \ldots$, then

$$f_n(s) \backsim s^n\ L(1/s) \quad (s \to o)$$

if and only if $g_n(s) \backsim s^n\ L(1/s)\ (m-m^{\alpha})$ $(s \to o)$

<u>Proposition(3.2.5)</u>:- If $\int_x^{\infty} L(t)\ dt/t$ converges, then $f_1(s) \backsim$ $sL(1/s)$ $(s \to o)$ implies

$$g_1(s) \backsim m\mu \int_{s^{-1}}^{\infty} L(t)\ dt/t/\log(1/m) \quad (s \to o).$$

<u>Proposition(3.2.6)</u>: If $n=1,2, \ldots,$ and $n < \alpha < n+1$, then

$$\int_o^1 f_n(s)\ L(1/s)\ ds/s^{1+\alpha} < \infty$$

if and only if

$$\int_o^1 g_n(s)\ L(1/s)\ ds/s^{1+\alpha} < \infty,$$

<u>Proposition(3.2.7)</u>:- (i) If n=2, 3, ...,

$$\int_0^1 f_n(s)\, L(1/s)\, ds/s^{n+1} < \infty$$

if and only if

$$\int_0^1 g_n(s)\, L(1/s)\, ds/s^{n+1} < \infty$$

(ii) If n=2, 3,....., and L is such that $\int_0^1 L(t)\,dt/t$ Converges, then

$$\int_0^1 f_n(s)\, L(1/s)\, ds/s^{n+2} < \infty$$

if and only if

$$\int_0^1 g_n(s)\, L(1/s)\, ds/s^{n+2} < \infty .$$

<u>Remark</u>:

It should be noted that all the implications from the conditioned limits to Z_1 follow directly from (3.2.11). However, for the sake of convinience, they are proved using (3.2.12).

R E F E R E N C E S

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